Lesson: Classification of Second Order Partial Differential Equations

Course Developer: Sada Nand Prasad

College/Department: Acharya Narendra Dev College

Table of Contents:

1. Learning Outcomes:

After studying this chapter, you will be able to

- classify linear second order PDEs into elliptic, parabolic and hyperbolic types;
- reduce linear second order PDEs into canonical form;
- classify linear equation with constant coefficient and *Euler equation*;
- obtain the general solution of linear second order PDEs;
- further Simplification of the reduced linear second order PDEs by introducing the new dependent variable;
- derive the Cauchy problem;

2. Introduction:

In the last chapter we have derived the three fundamental equations of mathematical physics, namely one and two - dimensional wave equation, one - dimensional heat conduction equation/diffusion equation and Laplace’s equation and mentioned that these equations are of hyperbolic, parabolic and elliptic type. Here, in this chapter we will explain in detail, about the classification of these second order partial differential equations with variable and constant coefficients and will find the general solution after reducing these equations to their respective canonical form.

To begin with, we have in this chapter described the second order partial differential equations (PDEs) in two independent variables and classified linear PDEs of second order into elliptic, parabolic and hyperbolic types.

3. Second – Order Partial Differential Equation in Two Independent Variables :

It is seen that a large number of PDEs arising in the study of applied mathematics with special reference to biological, physical and engineering applications, can be treated as a particular case of the most general form of a linear, second order PDE in two independents of the form

$$a\frac{\partial^2 z}{\partial x^2}+b\frac{\partial^2 z}{\partial x\,\partial y}+c\frac{\partial^2 z}{\partial y^2}+d\frac{\partial z}{\partial x}+e\frac{\partial z}{\partial y}+f\,z=g, \qquad \text{or,}$$

$$a\,z_{xx}+b\,z_{xy}+c\,z_{yy}+d\,z_x+e\,z_y+f\,z=g, \tag{1}$$

where a, b, c, d, e, f, and g are functions of the independent variables x and y and do not vanish simultaneously.

Value Addition: Do you know?
From coordinate geometry, we know that general equation of second degree in two variables $$a\,x^2+b\,x\,y+c\,y^2+d\,x+e\,y+f\,=0,$$ represents, ellipse if $b^2 - 4ac < 0$, parabola if $b^2 - 4ac = 0$ or, hyperbola if $b^2 - 4ac > 0$. (2)

The classification of second order partial differential equations (3.1), is suggested by the classification of the above equation (2), and based upon the possibilities of transforming equation (1) to canonical form at any point (x_0, y_0) by suitable coordinate transformation. Therefore, an equation at any point (x_0, y_0) is called

$$\begin{aligned}&\text{elliptic if } b^2(x_0, y_0) - 4\,a(x_0, y_0)\,c(x_0, y_0) < 0,\\ &\text{parabolic if } b^2(x_0, y_0) - 4\,a(x_0, y_0)\,c(x_0, y_0) = 0 \text{ or,}\\ &\text{hyperbolic if } b^2(x_0, y_0) - 4\,a(x_0, y_0)\,c(x_0, y_0) > 0.\end{aligned} \tag{3}$$

Example 1. Consider the equation, $z_{xx} + x^2 z_{yy} = 0,$, comparing this equation with equation (1), we get, a = 1, b = 0, c = x^2, and therefore, b^2- 4ac < 0***.*** Thus, the given equation is elliptic. Similarly, we can say that the Laplace's equation derived in the last chapter $u_{xx} + u_{yy} = 0,$ is elliptic.

Example 2. Consider the equation, $z_{xx} + 2z_{xy} + z_{yy} = 0,$, comparing this equation with equation (1), we get, a = 1, b = 2, c = 1, and therefore, b^2- 4ac = 0***.*** Thus, the given equation is parabolic. Similarly, we can say that the one – dimensional heat equation / diffusion equation derived in the last chapter $T_t = K T_{xx}$ is hyperbolic.

Example 3. Consider the equation, $z_{xx} = x^2 z_{yy}$, comparing this equation with equation (1), we get, a = 1, b = 0, c = x^2, and therefore, b^2- 4ac = $4x^2$ > 0***.*** Thus, the given equation is hyperbolic. Similarly, we can say that the one - dimensional wave equation derived in the last chapter $u_{tt} = c^2 u_{xx}$ is hyperbolic.

It may also be remarked here that an equation can be of mixed type, depending upon its coefficients.

Example 4. The equation given by $x z_{xx} + z_{yy} = x^2,$ can be classified as elliptic if, x > 0, parabolic, if x = 0, or hyperbolic if x < 0 as b^2- 4ac = - 4 x.

Now, we shall show that by a suitable change in the independent variables, we can reduce any equation of type (1) to one of the three *standard or canonical forms*. Let us suppose we change the variables x and y to u and v, respectively where

u = u (x, y), v = v(x, y). (4)

We will also assume that *u and v* are twice continuously differentiable and in the region under consideration the Jacobian

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \neq 0.$$

Then for the system (4), we can determine x and y uniquely. Suppose x and y are twice continuously differentiable functions of *u and v*. Then, we have,

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = z_u.u_x + z_v.v_x$$
$$z_y = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = z_u.u_y + z_v.v_y$$
$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)$$
$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u.\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$
$$= z_{uu}\, u_x^2 + 2\, z_{uv}\, u_x v_x + z_{vv}\, v_x^2 + z_u\, u_x^2 + z_v\, v_x^2$$
$$z_{xy} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)$$
$$= \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u.\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y \partial x} \tag{5}$$
$$= z_{uu}\, u_x u_y + z_{uv}\left(u_x v_y + u_y v_x\right) + z_{vv}\, v_x v_y + z_u\, u_{xy} + z_v\, v_{xy}$$
$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)$$
$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u.\partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}$$
$$= z_{uu}\, u_y^2 + 2\, z_{uv}\, u_y v_y + z_{vv}\, v_y^2 + z_u\, u_y^2 + z_v\, v_y^2$$

Substituting all these values in equation (1), we get,

$$a^* z_{uu} + b^* z_{uv} + c^* z_{vv} + d^* z_u + e^* z_v + f^* z = g^*$$

where

$$\begin{aligned}
a^* &= a u_x^2 + b u_x u_y + c u_y^2, \\
b^* &= 2 a u_x v_x + b\left(u_x v_y + u_y v_x\right) + 2c u_y v_y, \\
c^* &= a v_x^2 + b v_x v_y + c v_y^2, \\
d^* &= a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y, \\
e^* &= a v_{xx} + b v_{xy} + c v_{yy} + d v_x + e v_y, \\
f^* &= f, \\
g^* &= g.
\end{aligned} \tag{6}$$

The classification of the second order PDE (1) depends on the coefficients a(x, y), b(x, y), and c(x, y) at any point (x, y). So, we can write equation (1) and equation (2) as

$$a\, z_{xx} + b\, z_{xy} + c\, z_{yy} = h\left(x, y, z, z_x, z_y\right) \tag{7}$$

and

$$a^* z_{uu} + b^* z_{uv} + c^* z_{vv} = h^* \left(u, v, z, z_u, z_v \right) \qquad (8)$$

respectively.

4. Canonical Forms:

Now the problem is to determine *u* and *v* so that the equation (7) takes the simplest form. We first suppose that a ≠ 0, b ≠ 0, c ≠ 0. We will choose the new variables u and v such that the coefficients a^* and c^* in equation (6) vanish.

So, we get,

$$a^* = a u_x^2 + b u_x u_y + c u_y^2 = 0,$$
$$c^* = a v_x^2 + b v_x v_y + c v_y^2 = 0,$$

Or, equivalently,

$$a w_x^2 + b w_x w_y + c w_y^2 = 0, \qquad (9)$$

Dividing the above equation (9) throughout by w_y^2, we get,

$$a \left(\frac{w_x}{w_y} \right)^2 + b \left(\frac{w_x}{w_y} \right) + c = 0, \qquad (10)$$

If, $\lambda = \frac{w_x}{w_y}$, then eqn. (10) can be rewritten as

$$a \lambda^2 + b \lambda + c = 0, \qquad (11)$$

Equation (11) is a quadratic equation and its roots, are known as *characteristic equation*, are ordinary differential equation in the xy – plane along which *u* = constant and *v* = constant and the integrals of the roots are known as *characteristic curves*.

Value Additions: Do you know?
The procedure to determine *u* and *v* to get equation (7) in simplest form is simple when the discriminant $b^2 - 4ac$ of the quadratic eqn. (11) is positive, negative or zero.

Now we will discuss all these three cases one by one.

Case I.

If $b^2 - 4ac > 0$, then the roots λ_1, λ_2 of eqn. (11) are real and distinct. The coefficients a* and c* in eqn. (6) will vanish if we choose the new variables *u* and *v* such that

$$u_x = \lambda_1 u_y, \tag{12}$$

$$v_y = \lambda_2 v_y \tag{13}$$

Equations (12) and (13) will determine the form of *u* and *v* as a function of x and y and to determine this use Lagrange's auxiliary equation, that we had already discussed in the previous chapter.
The Lagrange's auxiliary equation of equation (12) is given by

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{du}{0}$$

du = 0 => *u* =constant and from the first two members, we have,

$$\frac{dy}{dx} + \lambda_1 = 0, \tag{14}$$

Suppose $\phi_1(x, y)$ = constant is the solution of the above equation (14), then the solution of equation (12) can be taken as,

$$u = \phi_1(x, y) \tag{15}$$

Similarly, if $\phi_2(x, y)$ = constant is the solution of the equation

$$\frac{dy}{dx} + \lambda_2 = 0, \tag{16}$$

then the solution of equation (13) can be taken as,

$$v = \phi_2(x, y) \tag{17}$$

In general, we can also observe that

$$4a^* c^* - b^{*2} = (4ac - b^2)(u_x v_y - u_y v_x)^2$$

so that when a* and c* are zero, then

$$b^{*2} = (b^2 - 4ac)(u_x v_y - u_y v_x)^2 \tag{18}$$

This follows that $b^{*2} > 0, (\because b^2 - 4ac > 0)$ and therefore we can divide both sides of the equation by it.
Making the substitution defined by equation (15) and equation (17), equation (7) transform to the form

$$z_{uv} = \phi(u, v, z, z_u, z_v) \tag{19}$$

Equation (19) is the *canonical or standard form* in this case.

Example 5: Reduce the equation in example 3, given by $z_{xx}=x^2 z_{yy}$ to *canonical form.*

From example 3, we know that the equation is hyperbolic and the quadratic equation (11) becomes $\lambda^2-x^2=0 \Rightarrow \lambda=\pm x$. Also, the equations (14) and (16) becomes

$$\frac{dy}{dx}+x=0 \text{ and } \frac{dy}{dx}-x=0.$$

Integrating both these equations, we get,

$$y+\frac{x^2}{2}=c_1 \text{ and } y-\frac{x^2}{2}=c_2 \Rightarrow u=y+\frac{x^2}{2} \text{ and } v=y-\frac{x^2}{2}$$

From equation (5), we get,

$$z_x=x(z_u-z_v),$$
$$z_y=z_u+z_v,$$
$$z_{xx}=x^2\left(z_{uu}-2z_{uv}+z_{vv}\right)+z_u-z_v,$$
$$z_{yy}=\ z_{uu}+2z_{uv}+z_{vv},$$

Substituting these values in the equation $z_{xx}=x^2 z_{yy}$, we get,

$$x^2\left(z_{uu}-2z_{uv}+z_{vv}\right)+z_u-z_v-x^2\left(z_{uu}+2z_{uv}+z_{vv}\right)=0,$$
$$z_{uv}=\frac{1}{4x^2}(z_u-z_v) \Rightarrow z_{uv}=\frac{1}{4(u-v)}(z_u-z_v),$$

This is the required canonical form of the hyperbolic equation $z_{xx}=x^2 z_{yy}$.

Case II.

When $b^2-4ac=0$, then the roots λ_1, λ_2 of equation (11) are equal. We will define the function *u* as we have defined in the earlier case and take *v* to be the function of x and y, which is independent of *u*.

We have a* = 0 as in the last case. Also, since $b^2-4ac=0$, therefore, b* = 0. In the case, we also take c* ≠ 0, otherwise *v* would be a function of *u*. Dividing equation (8) throughout by c*, we get,

$$z_{vv}=\phi(u,v,z,z_u,z_v) \qquad (20)$$

Equation (20) is called the *canonical form* of the *parabolic equation*.

Example 6: Consider the equation given in example 2, $z_{xx}+2z_{xy}+z_{yy}=0$, We want to reduce this parabolic equation to the canonical form. For this equation the quadratic equation (11) becomes $\lambda^2+2\lambda+1=0 \Rightarrow \lambda=-1,-1$. Also, the equation $\frac{dy}{dx}+\lambda=0$ becomes $\frac{dy}{dx}-1=0$. And integrating the equation, we get, *x* – *y* = *const*ant and consequently,

$u = x - y$. If we take $v = x + y$, which is a function of variables x and y independent of u, then from equation (5), we get,

$$z_x = z_u + z_v,$$
$$z_y = -z_u + z_v,$$
$$z_{xx} = z_{uu} + 2z_{uv} + z_{vv},$$
$$z_{xy} = -z_{uu} + z_{vv},$$
$$z_{yy} = z_{uu} - 2z_{uv} + z_{vv},$$

Substituting the values into the equation $z_{xx} + 2z_{xy} + z_{yy} = 0,$ we get,

$$z_{uu} + 2z_{uv} + z_{vv} - 2z_{uu} + 2z_{vv} + z_{uu} - 2z_{uv} + z_{vv} = 0,$$
$$\Rightarrow 4z_{vv} = 0, or\ z_{vv} = 0$$

which is the required *canonical form* of the equation $z_{xx} + 2z_{xy} + z_{yy} = 0$.

Case III.

For elliptic type equation, we have $b^2 - 4ac < 0,$ in this case the roots λ_1, λ_2 of equation (11) are complex. Proceeding, as in the case I, we find the equation (7) reduces to the form (19) but the variables *u* and *v* complex conjugate.

To get a real canonical form, we take $u = \alpha + i\beta,\ v = \alpha - i\beta$, so that

$$\alpha = \frac{1}{2}(u+v),\ \beta = \frac{1}{2}(u-v),$$

Now, $z_u = z_\alpha \alpha_u + z_\beta \beta_u = \frac{1}{2}\left(z_\alpha - i z_\beta\right)$

Similarly, we have,

$$z_v = z_\alpha \alpha_v + z_\beta \beta_v = \frac{1}{2}\left(z_\alpha + i z_\beta\right)$$
$$\therefore z_{uv} = \left(z_v\right)_u = \frac{1}{4}\left(z_{\alpha\alpha} + z_{\beta\beta}\right)$$

Again, transforming the variables *u* and *v* to α and β, respectively, we get

$$z_{\alpha\alpha} + z_{\beta\beta} = \phi\left(\alpha, \beta, z, z_\alpha, z_\beta\right) \qquad (21)$$

Equation (21) is called the *canonical form* of the *elliptic equation*.

Example 7: In this example we will reduce the elliptic equation in example 1, given by $z_{xx}+x^2 z_{yy}=0,$ to *canonical form.*

In this case the quadratic equation (11) becomes $\lambda^2+x^2=0 \Rightarrow \lambda=\pm i\,x$. Also, the equations (14) and (16) becomes

$$\frac{dy}{dx}+i\,x=0 \text{ and } \frac{dy}{dx}-i\,x=0$$

Integrating both these equations, we get,

$$y+i\frac{x^2}{2}=c_1 \text{ and } y-i\frac{x^2}{2}=c_2 \text{ => } u=y+i\frac{x^2}{2}=\alpha+i\,\beta \text{ and } v=y-i\frac{x^2}{2}=\alpha-i\,\beta,$$

$$\Rightarrow \alpha=y, \qquad \beta=\frac{x^2}{2}$$

From equation (5), we get,

$$z_x=z_\alpha.\alpha_x+z_\beta.\beta_x=x\,z_\beta,$$

$$z_y=z_\alpha.\alpha_y+z_\beta.\beta_y=z_\alpha,$$

$$z_{xx}=\left(z_x\right)_x=\left(x\,z_\beta\right)_x=z_\beta+x\left[\left(z_\beta\right)_x.\alpha_x+\left(z_\beta\right)_\beta.\beta_x\right]=z_\beta+x^2 z_{\beta\beta},$$

$$z_{yy}=\left(z_y\right)_y=z_{\alpha\alpha},$$

Substituting the values into the equation $z_{xx}+x^2 z_{yy}=0,$ we get,

$$z_\beta+x^2 z_{\beta\beta}+x^2 z_{\beta\beta}=0, \Rightarrow z_{\alpha\alpha}+z_{\beta\beta}=-\frac{1}{x^2}z_\beta,$$

$$or,\, z_{\alpha\alpha}+z_{\beta\beta}=-\frac{1}{2\alpha}z_\beta$$

which is the required canonical form of the equation $z_{xx}+x^2 z_{yy}=0$.

Now in the next section, we will discuss the canonical form of the second order PDE with constant coefficients, in the similar manner we have discussed for the second – order PDE with variable coefficients in this section.

5. Equations with Constant Coefficients:

When the coefficients of the equation (1), are constant and real, then the discriminant b^2-4ac is constant and therefore the equation will be of a single type at all points.

The general form of the second order PDE with constant coefficients is given by

$$a\,z_{xx}+b\,z_{xy}+c\,z_{yy}+d\,z_x+e\,z_y+f\,z=g\left(x,y\right), \qquad (22)$$

In particular, the equation given by

$$a\,z_{xx}+b\,z_{xy}+c\,z_{yy}=0, \qquad (23)$$

is called the *Euler equation*.

Now we will discuss all the three cases one by one as we have discussed in the case of equations with variable coefficients.

Case I. (Hyperbolic Type)

When $b^2-4ac>0,$ we get two distinct roots and integrating equation (14) and equation (16), we get,

$$y=-\lambda_1 x+c_1, \qquad y=-\lambda_2 x+c_2, \tag{24}$$

This implies,

$$u=y+\lambda_1 x, \qquad v=y+\lambda_2 x, \tag{25}$$

where

$$\lambda_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{26}$$

Equation (24), are two families of straight lines.

By using equation (25), equation (22) becomes

$$z_{uv}=d_1 z_u+e_1 z_v+f_1 z+g_1(u,v), \tag{27}$$

where d_1, e_1, f_1 are constant.

If in equation (26), $a = 0$, from equation (11), equation (14) and equaiton (16), we get,

$$a+\frac{b}{\lambda}+\frac{c}{\lambda^2}=0 \;\Rightarrow\; b\left(-\frac{dx}{dy}\right)+c\left(-\frac{dx}{dy}\right)^2=0 \Rightarrow -b\left(\frac{dx}{dy}\right)+c\left(\frac{dx}{dy}\right)^2=0,$$

=> $\dfrac{dx}{dy}=0,$ and $-b+c\left(\dfrac{dx}{dy}\right)=0,$

Integrating the above eqn. we get,

$$x=c_1 \quad and \quad x=\left(\frac{b}{c}\right)y+c_2,$$

where c_1 and c_2 are constants.

This implies,

$$u = x \quad and \quad v = x - \left(\frac{b}{c}\right) y, \tag{28}$$

Using equation (28), equation (22) reduces to the *canonical form*

$$z_{uv} = d_1^* z_u + e_1^* z_v + f_1^* z + g_1^*(u,v), \tag{29}$$

where d_1^*, e_1^* and f_1^* are constants.

For *Euler equation* given by equation (23), the canonical form is given by

$$z_{uv} = 0, \tag{30}$$

We can get the general solution of the *Euler equation* by integrating equation (30), and the solution is given by

$$z = \phi_1(u) + \phi_2(v) = \phi_1(y + \lambda_1 x) + \phi_2(y + \lambda_2 x), \tag{31}$$

where ϕ_1 and ϕ_2 are arbitrary functions, and λ_1 and λ_2 are given by equation (25).

Example 8: Consider the equation $z_{xx} + z_{xy} - 2z_{yy} - 3z_x - 6z_y = 9(2x - y)$, to reduce this equation to *canonical form*; first we will compare this equation with the equation (22) and get, *a = 1, b = 1, c = -2, d = - 3, e = - 6 and g = 9 (2 x - y).* This implies $b^2 - 4ac = 1 + 8 = 9 > 0$, and consequently the equation is hyperbolic.
Also, from equation (25) and equation (26), we get,

$$u = y + x, \qquad v = y - 2x, \qquad \because \lambda_{1,2} = \frac{-1 \pm \sqrt{9}}{2} = 1, -2$$

This gives,

$$\begin{aligned}
z_x &= -2z_u + z_v, \\
z_y &= z_u + z_v, \\
z_{xx} &= 4z_{uu} - 4z_{uv} + z_{vv}, \\
z_{xy} &= -2z_{uu} - z_{uv} + z_{vv}, \\
z_{yy} &= z_{uu} + 2z_{uv} + z_{vv},
\end{aligned}$$

Substituting these values into the equation,

$$z_{xx} + z_{xy} - 2z_{yy} - 3z_x - 6z_y = 9(2x - y),$$

we get,

$$4z_{uu} - 4z_{uv} + z_{vv} - 2z_{uu} - z_{uv} + z_{vv} - 2z_{uu} - 4z_{uv} - 2z_{vv} + 6z_u - 3z_v - 6z_u - 6z_v = -9u,$$

$$\Rightarrow z_{uv} + z_v = u,$$

which is the required *canonical form*.

Case II. (Parabolic Type)

When $b^2 - 4ac = 0,$ we get equal roots

$$\lambda_1 = \lambda_2 = \frac{-b}{2a}$$

and only one family of characteristic exists.

Also, $\frac{dy}{dx} + \lambda_1 = 0,$ => $\frac{dy}{dx} = -\lambda_1,$ and integration gives,

$$y + \lambda_1 x = c_1.$$

Let

$$u = y + \left(\frac{-b}{2a}\right)x$$

and $\quad v = hx + ky \qquad (32)$

where *h* and *k* are choose so that the transformation is not equal to zero.
Using equation (32), equation (22) reduces to the *canonical form*

$$z_{vv} = d_2^* z_u + e_2^* z_v + f_2^* z + g^*{}_2(u,v), \qquad (33)$$

If b = 0, we have 4 ac = 0 => a = 0 or c = 0. Then we observe that the equation is already in the canonical form. In the similar way if a = 0 or c = 0, then b = 0, and the equation is already in the canonical form in the parabolic equation, and the canonical form is given by,

$$z_{vv} = 0, \qquad (34)$$

Integrating the above equation, we get the general solution as

$$z_v = \phi_1(u) \Rightarrow u = v\phi_1(u) + \phi_2(u) \qquad (35)$$

Example 9: Let us reduce the equation $z_{xx} - 2z_{xy} + z_{yy} = 0,$ to *canonical form*, comparing this equation with the equation (22) we get, *a = 1, b = -2, c = 1, d =0, e =0 and g =0,* this gives $b^2 - 4ac = 4 - 4 = 0,$
and therefore the equation is parabolic.
From, equation (32), we have,

$$u = y + x, \text{ and } v = y,$$

chosen so that the transformation is not equal to zero.
This gives,

$$z_x = z_u,$$

$$z_y = z_u + z_v,$$

$$z_{xx} = z_{uu},$$

$$z_{xy} = z_{uu} + z_{uv},$$
$$z_{yy} = z_{uu} + 2z_{uv} + z_{vv},$$

Substituting these values, equation (22) reduces to the *canonical form*

$$z_{uu} - 2z_{uu} - 2z_{uv} + z_{uu} + 2z_{uv} + z_{vv} = 0,$$
$$\Rightarrow z_{vv} = 0.$$

Case III. (Elliptic Type)

When $b^2 - 4ac < 0$, the characteristics are complex conjugate and are given by

$$y = -\lambda_1 x + c_1, \qquad y = -\lambda_2 x + c_2, \tag{36}$$

where $\lambda_1 = A + iB, \lambda_2 = A - iB, c_1 \, and \, c_2$ are complex numbers and $A = \dfrac{-b}{2a}, B = \dfrac{\sqrt{4ac - b^2}}{2a}$

This implies,

$$u = y + \lambda_1 x, \qquad v = y + \lambda_2 x, \tag{37}$$

$$\begin{aligned} \alpha &= \frac{1}{2}(u+v) = \frac{1}{2}\big(y + (A+iB)x + y + (A-iB)x\big) = y + Ax, \\ \beta &= \frac{1}{2i}(u-v) = Bx \end{aligned} \tag{38}$$

Using equation (38), equation (22) reduces to the *canonical form*

$$z_{\alpha\alpha} + z_{\beta\beta} = d_2^* z_\alpha + e_2^* z_\beta + f_3^* z + g_3^*(\alpha, \beta), \tag{39}$$

Example 10: To reduce the equation $z_{xx} + z_{yy} = 0,$ to *canonical form*, we compare this equation with the equation (22) and we get, *a = 1, b = 0, c = 1, d =0, e =0 and g =0,* this gives $b^2 - 4ac = -4 < 0,$
and therefore the equation is elliptic.

From, equation (37), we have, $u = x + iy, v = x - iy,$ and therefore from equation (38), we get,

$$\alpha = \frac{1}{2}(u+v) = x, \; and \; \beta = \frac{1}{2i}(u-v) = y,$$

This gives,

$$z_x = z_\alpha,$$
$$z_y = z_\beta,$$
$$z_{xx} = z_{\alpha\alpha},$$
$$z_{yy} = z_{\beta\beta},$$

Substituting these values, equation (22) reduces to the *canonical form.*

$$z_{\alpha\alpha} + z_{\alpha\alpha} = 0.$$

6. General Solutions:

The general solution of the second order PDE is the most general possible relation between the independent variables x, y and the dependent variable z such that the value of z and the associated derivatives obtained from it and substituted in the given equation reduce it to an identity. Generally, it is difficult to determine the general solution of any given equation but reduction to canonical form simplifies the equation and then it becomes easier to get the general solution. If the canonical form is simple, then we can determine the general solution immediately.

Example 11: In example 8 we have reduced the equation $z_{xx} + z_{xy} - 2z_{yy} - 3z_x - 6z_y = 9(2x - y)$, to *canonical form*; let us determine the general solution of this equation.

Substituting $t = z_v$, in the canonical form $z_{uv} + z_v = u$, we get,

$$t_u + t = u,$$

I.F. = $e^{\int du} = e^u$

$$\Rightarrow t.e^u = \int u e^u + f(v),$$
$$\Rightarrow t = (u-1) + e^{-u} f(v),$$
$$\Rightarrow z_v = (u-1) + e^{-u} f(v),$$
$$\Rightarrow z(u,v) = (u-v)v + e^{-u} f(v) + g(v),$$
$$\Rightarrow z(u,v) = (x+y)(y-2x-1) + e^{(2x-y)} f(x+y) + g(y-2x).$$

Example 12: In example 9, we reduced the parabolic equation $z_{xx} - 2z_{xy} + z_{yy} = 0$, to *canonical form* $z_{vv} = 0$, in this example we will determine the general solution of this equation.

Integrating the equation $z_{vv} = 0$, we get,

$$z_v = f(u) \Rightarrow z = v f(u) + g(u),$$
$$\Rightarrow z(x,y) = y f(y+x) + g(y+x).$$

Example 13: In example 10, we have reduced the elliptic equation $z_{xx}+z_{yy}=0$, to *canonical form*; let us determine the general solution of this equation.

The canonical form is given by, $z_{\alpha\alpha}+z_{\alpha\alpha}=0$.We will first transform this equation in the independent variable *u* and *v* and to get this we have, $u=x+iy,\ v=x-iy,$ This gives,

$$z_x=z_u+z_v,$$
$$z_y=iz_u-iz_v,$$
$$z_{xx}=z_{uu}+2z_{uv}+z_{vv},$$
$$z_{yy}=-z_{uu}+2z_{uv}-z_{vv},$$

Substituting all these values in eqn. $z_{xx}+z_{yy}=0$, we get,

$$z_{uv}=0\Rightarrow z_u=f(u)\Rightarrow z=f(u)+g(v),$$
$$\therefore z(x,y)=f(x+iy)+g(x-iy).$$

7. Further Simplification:

We have already simplified the second order PDE with constant coefficients by reducing them to canonical forms. Here we will further simplify the reduced equations by introducing the new dependent variable

$$v=ze^{-(ar+bs)}, \tag{40}$$

Where a and b are undetermined coefficients.

We have classified the second order PDE with constant coefficients as

hyperbolic:

$$z_{rs}=a_1z_r+a_2z_s+a_3z+f_1 \tag{41}$$

$$z_{rr}-z_{ss}=a_1^*z_r+a_2^*z_s+a_3^*z+f_1^* \tag{42}$$

parabolic:

$$z_{ss}=b_1z_r+b_2z_s+b_3z+f_2 \tag{45}$$

elliptic:

$$z_{rr}+z_{ss}=c_1z_r+c_2z_s+c_3z+f_3 \tag{46}$$

where *r* and *s* are new variables in the transformations

$r = r(x, y)$, $s = s(x, y)$,
and the jacobian J ≠ 0.

From equation (40), we get,

$$z_r = (v_r + a v) e^{ar+bs},$$
$$z_s = (v_s + b v) e^{ar+bs},$$
$$z_{rr} = (v_{rr} + 2 a v_r + a^2 v) e^{ar+bs},$$
$$z_{rs} = (v_{rs} + a v_s + b v_r + a b v) e^{ar+bs},$$
$$z_{ss} = (v_{ss} + 2 b v_s + b^2 v) e^{ar+bs},$$

Substituting these values into equation (41), we get,

$$v_{rs} + (b - a_1) v_r + (a - a_2) v_s + (ab - a_1 b - a_2 b - a_3) v = f_1 e^{-(ar+bs)},$$

If $b = a_1$ and $a = a_2$ we get,

$$v_{rs} = (a_1 a_2 + a_3) v + g_1,$$

where

$$g_1 = f_1 e^{-(ar+bs)}.$$

Similarly by substituting these values into equation (42) – (44), we get the corresponding canonical forms as

$$\begin{aligned} &\text{hyperbolic:} && v_{rs} = h_1 v + g_1, \\ & && v_{rr} - v_{ss} = h_1^* v + g_1^*, \\ &\text{parabolic:} && v_{ss} = h_2 v + g_2, \\ &\text{elliptic:} && v_{rr} + v_{ss} = h_3 v + g_3, \end{aligned} \tag{45}$$

Example 14: Transform the equation $z_{xx} - z_{yy} + 3 z_x - 2 z_y + z = 0,$ to the form $v_{rs} = cv$, c = constant by introducing the new variable $v = z e^{-(ar+bs)}$, where a and b are undetermined coefficients.

Comparing this equation to equation (22), we get,

a = 1, b = 0, c = -1, d = 3, e = -2, f = 1 and g = 0, and
$b^2 - 4ac = 4 > 0,$ consequently the equation is hyperbolic.

Also, from equation (25) and equation (26), we get,

$$u = y - x, \qquad v = y + x, \qquad \because \lambda_{1,2} = \frac{0 \pm \sqrt{4}}{2} = 1, -1$$

This gives,

$$z_x = -z_u + z_v,$$
$$z_y = z_u + z_v,$$
$$z_{xx} = z_{uu} - 2z_{uv} + z_{vv},$$
$$z_{yy} = z_{uu} + 2z_{uv} + z_{vv},$$

Substituting these values into the equation, $z_{xy} - z_{yy} + 3z_x - 2z_y + z = 0,$ we get,

$$z_{uu} - 2z_{uv} + z_{vv} - z_{uu} - 2z_{uv} - z_{vv} - 3z_u + 4z_v - 2z_u - 2z_v + z = 0,$$
$$\Rightarrow -4z_{uv} - 5z_u + z_v + z = 0,$$

Substituting $v = ze^{-(ar+bs)}$, we have,

$$z_u = v_r e^{ar+bs} + vae^{ar+bs} = e^{ar+bs}(v_r + av),$$
$$z_{uv} = be^{ar+bs}(v_r + av) + e^{ar+bs}(v_{rs} + av),$$
$$z_v = v_s e^{ar+bs} + bve^{ar+bs},$$

Substituting these values in the eqn. $-4z_{uv} - 5z_u + z_v + z = 0$, we get,

$$-4\left[e^{ar+bs}\left(bv_r + bav + v_{rs} + av_s\right)\right] - 5\left[e^{ar+bs}\left(v_r + av\right)\right] + e^{ar+bs}\left(v_s + bv\right) + ve^{ar+bs} = 0,$$
$$\Rightarrow -4v_{rs} + (-4a+1)v_s + (-4b-5)v_r - 4abv - 5av + bv + v = 0,$$

Putting – 4 a + 1 = 0 => a = ¼, and – 4 b – 5 = 0 => b = - 5/4 in the above equation, we get,

$$-4v_{rs} - \frac{v}{4} = 0 \Rightarrow v_{rs} = \left(\frac{-1}{16}\right)v$$

which is the required form.

8. The Cauchy Problem:

Consider a second – order PDE for the function z, in the independent variables x and y, and suppose this equation can be solved explicitly for z_{yy} and hence it can be represented y the equation in the form

$$z_{yy} = F\left(x, y, z, z_x, z_y, z_{xx}, z_{xy}\right),$$

If the initial conditions are described along the same curve in the xy – plane, then this problem is called Cauchy problem.

Example 15: Consider the *Euler equation*

$$a z_{xx} + b z_{xy} + c z_{yy} = F\left(x, y, z, z_x, z_y\right), \tag{46}$$

where the coefficient a, b, c are function of x and y.

Let (x_0, y_0) be the point on the smooth curve in the xy – plane.

Let the parametric equation of this curve is

$$x_0 = x_0(\lambda),\ y_0 = y_0(\lambda) \tag{47}$$

where λ is a parameter.

Suppose two functions f (λ) and g (λ) are prescribed along the same curve and the Cauchy conditions are

$$z = f(\lambda) \tag{48 a}$$

$$z_n = g(\lambda) \tag{48 b}$$

where n is the direction of the normal to the curve.

For every point on the curve, every value of z is specified by equation (48 a). The curve represented by equation (47) with condition (48 a) yields a twisted curve in (x, y, z) space whose projection on the (x, y) is the curve we have considered.

Hence the solution of the Cauchy problem is the surface called an integral surface in the (x, y, z) space passing through the curve satisfying the condition (48 a) which represent a tangent plane to the integral surface along the curve.

If the function f (λ) is differentiable, then along the curve, we have,

$$\frac{dz}{d\lambda} = \frac{\partial z}{\partial x}\frac{dx}{d\lambda} + \frac{\partial z}{\partial y}\frac{dy}{d\lambda} = \frac{df}{d\lambda}, \tag{49}$$

$$\frac{\partial z}{\partial n} = \frac{\partial z}{\partial x}\frac{dx}{dn} + \frac{\partial z}{\partial y}\frac{dy}{dn} = g, \tag{50}$$

$$\frac{dx}{dn} = -\frac{dy}{ds} \qquad \& \qquad \frac{dy}{dn} = \frac{dx}{ds} \tag{51}$$

Putting equation (51) into equation (50), we get,

$$\frac{\partial z}{\partial n} = -\frac{\partial z}{\partial x}\frac{dy}{ds} + \frac{\partial z}{\partial y}\frac{dx}{ds} = g, \tag{52}$$

Since, $\begin{vmatrix} \dfrac{dx}{d\lambda} & \dfrac{dy}{d\lambda} \\ -\dfrac{dy}{ds} & \dfrac{ds}{dx} \end{vmatrix} = \dfrac{(dx)^2+(dy)^2}{ds\,d\lambda} \neq 0.$

Therefore we find z_x and z_y on the curve from the system of equations (49) and (52). Since, z_x and z_y are known on the curve, we can find higher derivatives by first differentiating z_x and z_y with respect to λ.

$$\frac{d}{d\lambda}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}\frac{dx}{d\lambda}+\frac{\partial^2 z}{\partial x\partial y}\frac{dy}{d\lambda} \qquad (53)$$

$$\frac{d}{d\lambda}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\partial y}\frac{dx}{d\lambda}+\frac{\partial^2 z}{\partial y^2}\frac{dy}{d\lambda} \qquad (54)$$

The given equation was

$$a\frac{\partial^2 z}{\partial x^2}+b\frac{\partial^2 z}{\partial x\partial y}+c\frac{\partial^2 z}{\partial y^2}=F, \qquad (55)$$

The equations (53) – (55) can be solved for z_{xx}, z_{xy} and z_{yy} if

$$\begin{vmatrix} \dfrac{dx}{d\lambda} & \dfrac{dy}{d\lambda} & 0 \\ 0 & \dfrac{dx}{d\lambda} & \dfrac{dy}{d\lambda} \\ a & b & c \end{vmatrix} \neq 0.$$

$$\frac{dx}{d\lambda}\left(c\frac{dx}{d\lambda}-b\frac{dy}{d\lambda}\right)-\frac{dy}{d\lambda}\left(-a\frac{dy}{d\lambda}\right)\neq 0,$$

$$\Rightarrow c\left(\frac{dx}{d\lambda}\right)^2-b\left(\frac{dx}{d\lambda}+\frac{dy}{d\lambda}\right)+a\left(\frac{dy}{d\lambda}\right)^2\neq 0,$$

$$\Rightarrow a\left(\frac{dy}{dx}\right)^2-b\left(\frac{dx}{d\lambda}+\frac{dy\,d\lambda}{(dx)^2}\right)+c\neq 0,$$

We know that

$$a\left(\frac{dy}{dx}\right)^2-b\left(\frac{dy}{dx}\right)+c=0,$$

is called the characteristic equation. Therefore the necessary condition for obtaining the second derivative is that the curve must not be characteristic curve.

Summary:

In this chapter we have covered the following

1) Classification of linear second order PDEs into hyperbolic, parabolic and elliptic types on the condition b^2 - 4ac greater than, equal to or less than zero.

2) We have discussed the canonical form of the second – order PDE variable and constant coefficients.

3) We have determined the general solution after reducing the PDE into its respective canonical forms.

4) We have further simplified the second order PDE with constant coefficients by reducing them to canonical forms by introducing the new dependent variable $v = z e^{-(ar+bs)}$.

5) Finally, we introduced the cauchy's problem.

Exercise:

1. Reduce the following equations in the canonical form after classifying them.

i. $y^2 z_{xx} - x^2 z_{yy} = 0,$

ii. $x^2 z_{xx} + 2xyu_{xy} + y^2 z_{yy} = 0,$

iii. $z_{xx} + x^2 z_{yy} = 0,$

iv. $4z_{xx} + 5z_{xy} + z_{yy} + z_x + z_y = 2,$

v. $z_{xx} - 4z_{xy} + 4z_{yy} = e^y,$

vi. $z_{xx} + z_{xy} + z_{yy} + z_x = 0,$

vii. $z_{xx} + 4z_{xy} + 4z_{yy} = 0,$

viii. $3z_{xx} + 4z_{xy} - \frac{3}{4} z_{yy} = 0,$

ix. $x z_{xx} + z_{yy} = x^2,$

x. $z_{xx} + z_{xy} - x z_{yy} = 0,$

2. Determine the general solution of the following equation.

i. $4z_{xx} + 5z_{xy} + z_{yy} + z_x + z_y = 2,$

ii. $3z_{xx} + 4z_{xy} - \frac{3}{4} z_{yy} = 0,$

iii. $y^2 z_{xx} - x^2 z_{yy} = 0,$

iv. $x^2 z_{xx} + 2xyu_{xy} + y^2 z_{yy} = 0,$

v $z_{xx} + x^2 z_{yy} = 0,$

3. Determine the general solution of the following equation.

i. $z_{xxxx} + 2z_{xxyy} + z_{yyyy} = 0,$

ii. $z_{xx} = x^2 z_{yy}$

4. Transform the equation $3z_{xx} + 7z_{xy} + 2z_{yy} + z_y + z = 0,$ to the form $v_{rs} = cv,\ \mathrm{c} = \mathrm{constant}$ by introducing the new variable $v = ze^{-(ar+bs)},$ where a and b are undetermined coefficients.

Glossary:

C Cauchy Problem: Consider a second – order PDE for the function z, in the independent variables x and y, and suppose this equation can be solved explicitly for z_{yy} and hence it can be represented y the equation in the form $z_{yy} = F\left(x, y, z, z_x, z_y, z_{xx}, z_{xy}\right),$ If the initial conditions are described along the same curve in the xy – plane, then this problem is called Cauchy problem.

D Discriminant : for the second order partial differential equation

$$a z_{xx} + b z_{xy} + c z_{yy} + d z_x + e z_y + f z = g, \tag{1}$$

the value $b^2 - 4ac$ is called the discriminant of the second order partial differential equation.

E Elliptic: Partial differential equation of second order represented by equation (1) is called elliptic iff $b^2 - 4ac < 0$.

H Hyperbolic: Partial differential equation of second order represented by equation (1) is called hyperbolic iff $b^2 - 4ac > 0$.

P Parabolic: Partial differential equation of second order represented by equation (1) is called parabolic iff $b^2 - 4ac = 0$.

References:

1. **Tyn Myint-U, Lokenath Debnath:** Linear Partial Differential Equations for Scientists and Engineers, Springer
2. **Sneddon, I.N.,** Elements of Partial Differential Equations, McGraw-Hill, New York (1957)
3. **Robert C. McOwen,** Partial Differential Equations Methods and Applications, Pearson
4. **Epstein B.,** *Partial Differential Equations,* McGraw- Hill, New York (1962).
5. **Hadamard J**., *Lectures on Cauchy's Problem in Linear Partial Differential Equations,* Dover Publications, New York (1952).